# Why Volume of a Cone is ⅓πr²h

An Intuitive, Visual & Calculus-Free Geometry Guide for S2 Students

No Calculus Required | Cavalieri's Slice Principle | S2 Mensuration Series

THE CONE MASTER FORMULA

**Volume of Cone = ⅓ × Base Area ($\pi r^2$) × Perpendicular Height ($h$)**

Where $r$ is the base circle radius and $h$ is the perpendicular distance from apex to base plane.

## 1 Linear Scaling of the Cone Radius ($t$ Scaling)

A circular cone is essentially a pyramid with a **circular base**.

When viewed from the side, a cone forms a triangle. As you move down a depth fraction $t$ from the apex ($t=0$ at top, $t=1$ at base):

> **Linear Radius Rule:** By similar triangles, the radius of a circular slice at depth $t$ grows linearly:
> $$\text{Radius of Slice} = t \cdot r$$

- At the Apex ($t = 0$): Radius = $0 \cdot r = 0$
- Halfway Down ($t = 0.5$): Radius = $0.5r$
- At the Base ($t = 1$): Radius = $1r = r$

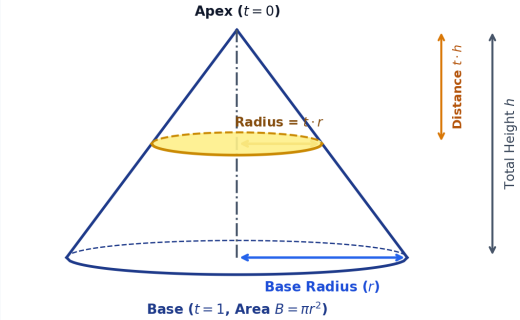


*Figure 1: Side view showing similar triangles. At depth fraction t, slice radius is tr.*

## 2 Quadratic Area Scaling ($t^2$ Scaling)

Area is 2-dimensional (***Length*** × ***Width***). Scaling radius by $t$ scales circular area by $t^2$:

$$\text{Slice Area} = \pi(tr)^2 = t^2(\pi r^2) = t^2 B$$

| Depth ($t$) | Slice Radius | Area Calculation | Slice Area relative to Base |
|---|---|---|---|
| $t = 0.00$ (Apex) | $0$ | $\pi(0)^2$ | **0** (Point) |
| $t = 0.25$ (¼ Down) | $\frac{1}{4} r$ | $\pi(\frac{1}{4} r)^2$ | $^1/_{16}$ **B** |
| $t = 0.50$ (Halfway) | $\frac{1}{2} r$ | $\pi(\frac{1}{2} r)^2$ | ¼ **B** |
| $t = 0.75$ (¾ Down) | $\frac{3}{4} r$ | $\pi(\frac{3}{4} r)^2$ | $^9/_{16}$ **B** |
| $t = 1.00$ (Base) | $r$ | $\pi r^2$ | **B** (Full Base) |

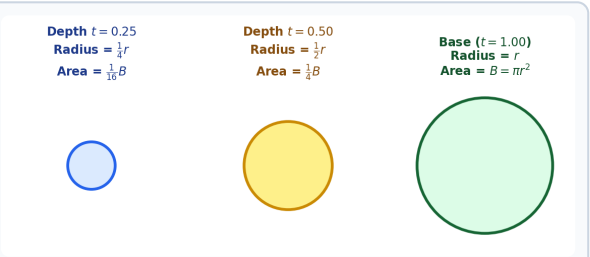


*Figure 2: Top-down views of circular slices. Area scales with the square of depth ($t^2$).*

## 3 Cavalieri Comparison: Cone vs. Pyramid

Place a **Cone** next to a **Square Pyramid** of identical height $h$ and matching base area $B$.

**Matching Slices at Every Height Level:**

- **Cone Slice Area at depth $t$: $t^2 \cdot B$**
- **Pyramid Slice Area at depth $t$: $t^2 \cdot B$**

Because their cross-sectional areas match at *every single level*, Cavalieri's Slice Principle guarantees their total volumes are **identical**!

$$V_{cone} = V_{pyramid} = \tfrac{1}{3} Bh = \tfrac{1}{3} \pi r^2 h$$

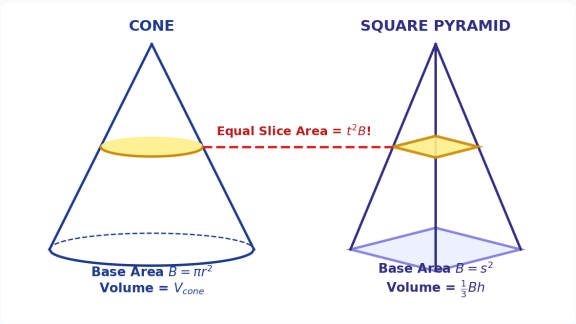


*Figure 3: Matching cross-sectional areas at all levels guarantee equal total volume.*

## 4 Direct Algebraic Comparison with Cube Reference

We can also compare the cone directly with our reference **Cube Pyramid** (Base = $4h^2$, Height = $h$, Volume = $(4/3)h^3$):

| Feature | Circular Cone | Reference Cube Pyramid | Ratio (Cone ÷ Reference) |
|---|---|---|---|
| **Height** | $h$ | $h$ | $1 : 1$ |
| **Base Area** | $B = \pi r^2$ | $4h^2$ | $B / 4h^2$ |
| **Slice Area at depth $t$** | $t^2 B$ | $t^2 (4h^2)$ | $(t^2 B) / (t^2 \cdot 4h^2) = B / 4h^2$ ($t^2$ cancels out!) |

**Final Calculation:**

$$V_{cone} = (B / 4h^2) \times V_{cube\ pyramid} = (B / 4h^2) \times ((4/3)h^3) = \tfrac{1}{3} Bh = \tfrac{1}{3} \pi r^2 h$$

## 🎓 Classroom Takeaway Summary

1. **Radius Scaling:** Depth fraction $t$ scales radius linearly ($tr$).
2. **Area Scaling:** Circular area scales quadratically ($t^2B$).
3. **The Cone-Cylinder Relationship:** A cone occupies exactly **one-third** the volume of a cylinder with the same base and height: $V = \tfrac{1}{3} \pi r^2 h$.